E-x-p-a-n-d-e-d Learning Progressions Frameworks for K-12 Mathematics: A Companion Document to the *Learning Progressions Frameworks Designed for Use with The Common Core State Standards in Mathematics K-12*

*Part 3 - Grades 9-12**

This document presents expanded views of the *Learning Progressions Frameworks for grades 9-12 Mathematics* to show how smaller learning progressions can be drawn from the Learning Progressions Framework (LPF) using the Progress Indicators and highlighted links to the Common Core State Standards (CCSS) in mathematics. At high school, grade levels are not broken down into specific grades, since specific courses will better determine which standards are taught and assessed. (*Two other expanded version documents are also available: Part 1 for elementary school/ grades K-4, and Part 2 for middle school/grades 5-8.) Each of these expanded versions displays all six LPF mathematics strands "unpacked" for ease of use by teachers and curriculum and assessment developers. The original document upon which they are based - *Learning Progressions Frameworks Designed for Use with the Common Core State Standards in Mathematics K-12,* including a more complete explanation of the research-based rationale and conceptual underpinnings - can be found at http://www.nciea.org/publications/Math_LPF_KH11.pdf *or at www.naacpartners.org*. Other support materials *and related publications can be found at www.nciea.org or at www.naacpartners.org*.

Karin K. Hess, Editor
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Expanded High School (Grades 9-12) Mathematics Strands

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Two other LPF expanded version mathematics documents are available for elementary school/ grades K-4, and middle school/grades 5-8. The original document upon which they are based - *Learning Progressions Frameworks Designed for Use with the Common Core State Standards in Mathematics K-12,* and other support materials *and related publications* can be found at *www.nciea.org*.

Important CCSS notes: All glossary, table, and footnote references embedded in the CCSS standards refer back to the original *Common Core State Standards for Mathematics* document which can be located at http://www.corestandards.org/the-standards/mathematics . Also note: '"Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (⋆)" (CCSS, p. 57).

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Overview: The Learning Progressions Frameworks (LPFs) Development Process

The approach used to identify the content progressions and specific standards within the *Common Core State Standards* (CCSS) considered three important dimensions. First, national content experts and researchers in mathematics were asked to identify specific content strands that represented a way to organize essential learning for all students, K-12. Next, the committee was asked to describe the "enduring understandings" (as defined by Wiggins and McTighe, 2005) for each particular content strand, as well as articulate what the learning targets would look like if students were demonstrating achievement of the enduring understandings at the end of each grade span (K-4, 5-8, and 9-12). The grade span learning targets for each strand are stated as broader performance indicators (e.g., Use equations and expressions involving basic operations to represent a given context; Build flexibility with whole numbers and fractions to understand the nature of number and number systems). The larger grained grade span learning targets are designed to describe progressively more complex demonstrations of learning across the grade spans and use wording similar to what one might see in performance level descriptors for a given grade or grade span. (See the original K-12 LPF document for all of the grade span learning targets - http://www.nciea.org/publications/Math_LPF_KH11.pdf.)

In mathematics, **six major LPF strands** were established. Below is a brief description of the six strands identified by the LPF content committee. "For each content area, these essential threads [strands] interact to build greater understanding of the discipline over time. Identifying a small number of essential threads makes the learning progression manageable for the classroom teacher in terms of tracking ongoing progress in the classroom" (Hess, 2008, p.5). It is not the intent that skills/concepts from a particular strand be taught in isolation in a linear sequence, but rather be integrated among strands, such as in a problem solving situation where students are demonstrating their understanding of measurement concepts while applying their knowledge of numbers and operations and using symbolic expression. *In other words, the LPFs should be thought of as a general map for learning, not a single route to a destination.*

- **Symbolic Expression (SE)** – Symbolic Expression, presented in this document as the first strand, is a reminder NOT to teach symbolic representations before students have begun to demonstrate conceptual understanding of what the symbols or procedures actually mean (e.g., what joining together (+) and taking apart (-) sets means; understanding relative magnitude of part-whole; that "equivalence" (=) means different names for the same number). Progress indicators for the Symbolic Expression strand should be taught in conjunction with skills and concepts described in PIs from other strands and introduced with building conceptual understanding in mind.
- **The Nature of Numbers & Operations (NO)** – The skills and concepts within the Nature of Numbers and Operations strands form the foundation - and often are the prerequisite skills and concepts - for many of the other mathematics strands. Local curriculum development efforts should always consider how the skills and concepts described in the Numbers & Operations progress indicators can be introduced, practiced, and extended with skills/concepts found in the other strands. The third N&O strand focuses on mathematical reasoning and problem solving. These progress indicators can be integrated with many CCSS

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standards at each grade level using problem solving contexts. While listed under the Nature of Numbers & Operations strand, the skills and concepts described in these progress indicators could apply to concepts in different mathematics strands, such as when developing proofs in Geometry.

- **Measurement (ME)** – Progress indicators are organized under two key learning targets for each grade span**.**
- **Patterns, Relations, & Functions (PFR)** – Progress indicators are organized under two key learning targets for each grade span.
- **Geometry (GM)** – Progress indicators are organized under one key learning target for each grade span.
- **Data Analysis, Probability, & Statistics (DPS)** – Progress indicators are organized under two key learning targets for each grade span. There is minimal emphasis in the CCSS on Data Analysis, Probability, & Statistics at grades K-6. Because many of the DPS mathematics skills and concepts are essential to science and social studies instruction at these grade levels, progress indicators are included in the DPS strand to guide unit development where organizing and interpreting data is important. However, you will not find many links to the CCSS mathematics standards in this strand at the lower grade levels.

Once the content committee had established the broader grade span learning targets for each strand, they were asked to identify and describe the essential skills and concepts needed to achieve the grade span expectations; use research syntheses to establish a general order of how those skills and concepts emerge for most students; and further break down the descriptors into smaller grades spans: K-2, 3-4, 5-6, 7-8, and high school. The descriptors of related skills and concepts became what we now call the **progress indicators** and the ordering/numbering used (1a, 1b, 1c, etc.) reflects the research base used to establish a general learning continuum. This means that descriptions of earlier skills build the foundation for later skills (e.g., later within a grade level, later at the next grade level/span).

The final step in the LPF development process was to identify alignment with specific CCSS mathematics content standards in order to create guidance for a cohesive curriculum experience across grades. Sometimes multiple standards from within the smaller grade spans could be linked to the same progress indicator (PI); sometimes there was only one or no standard that aligned. For example, in some strands and grade spans you will see PI descriptors that do not link (align) with an existing CCSS standard; however, the research review identified critical learning at certain stages during the learning process or skills that may be essential for conceptual understanding and for demonstrating progress. Therefore, progress indicators with no CCSS links are also included in the LPF to guide instruction, formative assessment, and progress monitoring.

Reading and Interpreting the LPF format, progress indicators, and related standards for a Grade Span – See next page. 

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***NO*: Understandings of number - "how many" or "how much" – and number types extend applications of arithmetic properties, operations, and number systems and guide use of computational strategies and algorithms.**

The statement of enduring understanding across grade spans states WHY learning the skills and concepts (and standards) listed below are important and how they are generally applied.

(K-4) Elementary School Learning Targets

E.NO-2 Build an understanding of computational strategies and algorithms:

- *Fluently add, subtract, multiply, divide, and estimate;*
- *Perform and represent operations with whole numbers, fractions, and mixed numbers;*
- *Identify multiples and factors of whole numbers.*

K-4 Elementary School Grade Span Learning Targets

- By the end of grade 4, students demonstrate and apply the skills and concepts related to Numbers & Operations in a variety of situations or problem solving contexts.
- Learning targets are the more general/broad performance descriptors associated with specific skills and concepts at each grade level.

Larger grade spans are then broken into smaller grade spans

Grades K-2	Grades 3-4
Build understanding and fluency with operations...	Build understanding and fluency with operations...
E.NO.2a representing addition and subtraction in multiple ways (composing/decomposing numbers, diagrams, using objects, arrays, equations, number lines), including regrouping K.OA-1, 2, 3, 4; K.NBT-1 1.OA-1, 2, 5, 6; 1.NBT-4, 5, 6 2.OA-1, 4; 2.NBT-7	E.NO.2d modeling multiplication (equal-sized groups, arrays, area models, equal-sized jumps on number lines, multiplicative comparisons) and division (successive subtraction, partitioning, sharing) of whole numbers 3.OA-1, 2, 3, 4, 5 4.OA-1, 2, 3; 4.NBT- 5, 6
E.NO.2b explaining or modeling the relationship between addition and subtraction 1.OA-3, 4 2.NBT-5, 7, 9	E.NO.2e describing relationships between addition-multiplication; multiplication-division; addition-subtraction; why commutativity does not apply to subtraction or division 3.OA-7, 9; 3.NBT-2 4.OA-2
E.NO.2c working flexibly with common addition and subtraction situations K. OA-2 1. OA-3, 5, 6, 8 2.OA-1, 2; 2.NBT- 2, 5, 7	E.NO.2f identifying factors and multiples of numbers 3.OA-6 4.OA-4
	E.NO.2g recognizing fractions as one number/one quantity, rather than two numbers (numerator and denominator) and using number lines to represent magnitude of fractions 3.NF-1, 2, 3a, 3c

What you see articulated in this sample LPF strand:

- "E" denotes all Elementary (K-4) progress indicators.
- Most LPF progress indicators are stated in a more general way (e.g., using many related strategies; using both addition and subtraction) than a single CCSS standard; therefore progress indicators (PIs) often align with several CCSS standards at different grade levels within the grade span. This multi-standard alignment can provide insights into potential "mini progressions" for lesson design.
- Numerous CCSS standards align with the first descriptor under K-2 and can be interpreted that this progress indicator embodies many important foundational skills for all three grade levels, K, 1, and 2. Teachers at all of these grades may need to revisit lower grade level skills (and standards) for students needing reinforcement/ extra work on prerequisite skills.
- K students would spend most of their school year working on CCSS standards: K.OA-1, 2, 3, 4; and K.NBT-1 (linked to the first PI), while grades 1 and 2 would be addressing all three PIs and the associated CCSS standards in this general/a-b-c order.

Text in blue denotes links to CCSS standards:
2.OA-1,2 means grade 2, Operations & Algebraic Thinking, standards 1 and 2
(See p. 19 of CCSS for mathematics)

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The highlighting in the expanded version of the LPF shows potential smaller learning progressions (LPs) and parts of the CCSS standards that link with progress indicators.

Elementary School Learning Targets

Data Analysis, Probability, and Statistics (DPS)- *DPS-1 Gather and interpret data to answer questions related to a particular/single context. Formulate questions, gather data, and build representations; Identify and describe variation in data, and describe and compare shapes of distributions and measures of central tendency.*

Progress Indicators for Grades K-2	Grade K CCSS standards	Grade 1 CCSS standards	Grade 2 CCSS standards
E.DPS.1a posing questions of interest that can be answered by counting or collecting data (e.g., concrete comparisons about students, classroom materials, science topics) *with teacher guidance*	**K.CC-5** **5. Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects.** **K.CC-6** **6. Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.[1]** **([1] Include groups with up to ten objects.)**	**1-MD-1** **1. Order three objects by length; compare the lengths of two objects indirectly by using a third object.**	**2.MD-2** **2. Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.** **2.MD- 5** **5. Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.**
E.DPS.1b identifying and sorting data/attributes; identifying rules for classifying data/attributes	**K.MD-1** **1. Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object.** **K.MD-2** **2. Directly compare two objects with a measurable attribute in common, to see which object has "more of"/"less of" the attribute, and describe the difference. *For example, directly compare the heights of two children and describe one child as taller/shorter.*** **K.MD-3** **3. Classify objects into given categories; count the numbers of objects in each category and sort the categories by count.[1]** **K.G-2** **2. Correctly name shapes regardless of their orientations or overall size.** **K.G-4** **4. Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts & attributes**	**1.MD-1** **1. Order three objects by length; compare the lengths of two objects indirectly by using a third object.** **1.MD-4** **4. Organize, represent, and interpret data with up to three categories; ask and answer questions about the total number of data points, how many in each category, and how many more or less are in one category than in another.** **1.G-1** **1. Distinguish between defining attributes (e.g., triangles are closed and three-sided) versus non-defining attributes (e.g., color, orientation, overall size); build and draw shapes to possess defining attributes.**	**2.G-1** **1. Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces.[1] Identify triangles, quadrilaterals, pentagons, hexagons, and cubes.** **2.MD-10** **10. Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems[1] using information presented in a bar graph.**

Highlighting indicates links among the Progress Indicator & one or more CCSS standard or parts of the standard(s).

The highlighting in 2 colors here illustrates two possible smaller learning progressions (LPs) for instruction & assessment for this PI. This document does not show multiple LPs with different highlighting for PIs, but different LPs may exist if you look for them by matching the highlighted CCSS language with the PI description.

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Expanded Learning Progressions Frameworks for K-12 Mathematics

High School Strands

Grades 9-12

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High School (Grades 9-12) Learning Targets, Progress Indicators, & Common Core Standards

Symbolic Expression (SE): The use and manipulation of symbols and expressions provide a variety of representations for solving problems and expressing mathematical concepts, relationships, and reasoning.

H.SE-1 *Represent relationships and interpret expressions and equations in terms of a given context (including complex and families of functions) for determining unknown values (including two or more variables): Represent and interpret multi-step problems; Represent complex numbers and vectors; Demonstrate the relationship between systems of equations and matrix representations; Represent the relationship between functions and modeling.*

Progress Indicators for Grades 9-12	**Grades 9-12 CCSS standards** Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★) (CCSS, p. 57).
H.SE.1a interpreting and using symbols to express relationships (e.g., identifying parts of expressions, generating equivalent expressions, formulas; exponents) A.SSE-1, 2, 3 F.IF-8b	**A.SSE-1, 2, 3** **1.** Interpret expressions that represent a quantity in terms of its context.★ a. Interpret parts of an expression, such as terms, factors, and coefficients. b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P.* **2.** Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.* **3.** Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. a. Factor a quadratic expression to reveal the zeros of the function it defines. b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines. c. Use the properties of exponents to transform expressions for exponential functions. *For example the expression 1.15^t can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.* **F.IF-8b** **8.** Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. b. Use the properties of exponents to interpret expressions for exponential functions. For example, identify percent rate of change in functions such as y = (1.02)t, y = (0.97)t, y = (1.01)12t, y = (1.2)t/10, and classify them as representing exponential growth or decay.
H.SE.1b creating mathematical models, using rules and relationships to describe and predict objects and events in the real world F.BF-1a S.ID-6a	**F.BF-1a** **1.** Write a function that describes a relationship between two quantities. a. Determine an explicit expression, a recursive process, or steps for calculation from a context. **S.ID-6a** **6.** Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.
H.SE.1c identifying essential quantitative relationships in a situation, using symbolic expressions to represent it, and drawing reasonable conclusions from it A.SSE-1,2, 3 G.GMD-3	**A.SSE-1, 2, 3** **1.** Interpret expressions that represent a quantity in terms of its context.★ a. Interpret parts of an expression, such as terms, factors, and coefficients. b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P.* **2.** Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.* **3.** Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. a. Factor a quadratic expression to reveal the zeros of the function it defines. b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

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	c. Use the properties of exponents to transform expressions for exponential functions. *For example the expression 1.15^t can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.* **G.GMD-3** **3.** Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.
H.SE.1d interpreting and using symbols to express relationships and solve problems (e.g., volume formulas; exponents, square and cube roots) F.IF-8b N.RN-1 G.GMD-3	**F.IF-8b** **8.** Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. b. Use the properties of exponents to interpret expressions for exponential functions. For example, identify percent rate of change in functions such as y = (1.02)t, y = (0.97)t, y = (1.01)12t, y = (1.2)t/10, and classify them as representing exponential growth or decay. **N.RN-1** **1.** Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5.* **G.GMD-3** **3.** Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.
H.SE.1e recognizing that there limitations in mathematics models A.CED-3 S.IC-2	**A.CE-3** **3.** Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* **S.IC-2** **2.** Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.

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High School (Grades 9-12) Learning Targets, Progress Indicators, & Common Core Standards

The Nature of Numbers and Operations (NO): Understandings of number - "how many" or "how much" – and number types extend applications of arithmetic properties, operations, and number systems and guide the use of computational strategies and algorithms.

H.NO-1 *Demonstrate flexibility using rational and irrational numbers and number systems, including complex numbers and matrices.*

Progress Indicators for Grades 9-12	**Grades 9-12 CCSS standards** Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★) (CCSS, p. 57).
H.NO.1a using exponents and scientific notation to represent quantities and expressions ***Also Addressed at Grade 8: 8.EE-1, 2, 3, 4*** **N.RN-2** **A.SSE-2, 3c**	(Grade 8) 8.EE-1, 2, 3, 4 **1.** Know and apply the properties of integer exponents to generate equivalent numerical expressions. For example, $3^2 \times 3^{-5} = 3^{-3} = 1/3^3 = 1/27$. **2.** Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that √2 is irrational. **3.** Use numbers expressed in the form of a single digit times a whole-number power of 10 to estimate very large or very small quantities, and to express how many times as much one is than the other. *For example, estimate the population of the United States as 3 times 10^8 and the population of the world as 7 times 10^9, and determine that the world population is more than 20 times larger.* **4.** Perform operations with numbers expressed in scientific notation, including problems where both decimal and scientific notation are used. Use scientific notation and choose units of appropriate size for measurements of very large or very small quantities (e.g., use millimeters per year for seafloor spreading). Interpret scientific notation that has been generated by technology. **N.RN-2** 2. Rewrite expressions involving radicals and rational exponents using the properties of exponents. **A.SSE-2, 3c** **2.** Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.* **3.** Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. c. Use the properties of exponents to transform expressions for exponential functions. *For example the expression 1.15^t can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*
H.NO.1b distinguishing rational numbers (terminating and repeating) from irrational numbers (non-terminating and non-repeating), and recognizing that together they form the real number system and that both can be represented on the number line ***Also Addressed at Grade 8: 8.NS-1, 2*** **N.RN-3**	**(Grade 8) 8.NS-1, 2** **1.** Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number. **2.** Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). *For example, by truncating the decimal expansion of √2, show that √2 is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.* **N.RN-3** **3.** Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.
H.NO.1c modeling and describing that complex numbers augment real numbers	*No specific Common Core Standards have been linked to this Progress Indicator at this grade level; however, these skills/concepts represent the "hypothesized" learning continuum.*
H.NO.1d extending operations and properties to work with complex numbers **N.CN-2**	**N.CN-2** **2.** Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.
H.NO.1e using matrices to store and manipulate data	*No specific Common Core Standards for all students have been linked to this Progress Indicator at this grade span; however, there are CCSS standards that address matrices, marked with "+" to indicate skills/concepts included in higher level math courses.*

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High School (Grades 9-12) Learning Targets, Progress Indicators, & Common Core Standards

The Nature of Numbers and Operations (NO): Understandings of number - "how many" or "how much" – and number types extend applications of arithmetic properties, operations, and number systems and guide the use of computational strategies and algorithms.

H.NO-2 *Build an understanding of computational strategies and algorithms including matrices and irrational and complex numbers: Use matrix operations and complex and irrational number operations; Apply exponential expressions (laws and properties).*

Progress Indicators for Grades 9-12	**Grades 9-12 CCSS standards** Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★) (CCSS, p. 57).
H.NO.2a using operations with rational numbers; representing rational numbers and approximations of irrational numbers on a number line **N.RN-3** **A.SSE-3b** **A.REI-2** **A.APR-1**	**N.RN-3** **3.** Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational. **A.SSE-3b** **3.** Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines. **A.REI-2** **2.** Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise. **A.APR-1** 1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.
H.NO.2b operating with irrational and complex numbers **A.REI-2** **N.RN-3** **N.CN-1, 2**	**A.REI-2** **2.** Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise. **N.RN-3** 3. Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational. **N.CN-1, 2** **1.** Know there is a complex number *i* such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real. **2.** Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.
H.NO.2c identifying exponential situations and applying the laws and properties of exponents in simplifying expressions and solving equations **A.SSE- 2, 3** **N.RN-1, 2**	**A.SSE-2, 3** **2.** Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.* **3.** Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★ a. Factor a quadratic expression to reveal the zeros of the function it defines. b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines. c. Use the properties of exponents to transform expressions for exponential functions. *For example the expression 1.15^t can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%* **N.RN-1, 2** **1.** Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5.* **2.** Rewrite expressions involving radicals and rational exponents using the properties of exponents.
H.NO.2d using matrices to represent	*No specific Common Core Standards for all students have been linked to this Progress Indicator at this grade span; however, there are CCSS standards*

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situations; perform and interpret basic matrix operations	*that address matrices, marked with "+" to indicate skills/concepts included in higher level math courses.*

High School (Grades 9-12) Learning Targets, Progress Indicators, & Common Core Standards

The Nature of Numbers and Operations (NO): Understandings of number - "how many" or "how much" – and number types extend applications of arithmetic properties, operations, and number systems and guide the use of computational strategies and algorithms.

H.NO-3 *Develop metacognitive skills through use of mathematical arguments to justify reasonableness of outcomes, to support formal proofs (including technology applications), and to develop metacognitive skills.*

Progress Indicators for Grades 9-12	Grades 9-12 CCSS standards Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★) (CCSS, p. 57).
H.NO.3a comparing the effectiveness of two plausible arguments, distinguishing correct logic or reasoning from that which is flawed, and if there is a flaw in an argument, explaining it **Functions** **F.IF-9** **F.LE-1a** **F.TF-8**	**Functions** **F.IF-9** **9.** Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.* **F.LE-1a** **1.** Distinguish between situations that can be modeled with linear functions and with exponential functions. a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals. **F.TF-8** **8.** Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find sin(θ), cos(θ), or tan(θ) given sin(θ), cos(θ), or tan(θ) and the quadrant of the angle.
H.NO.3a comparing the effectiveness of two plausible arguments, distinguishing correct logic or reasoning from that which is flawed, and if there is a flaw in an argument, explaining it **Number & Quantity** **N.RN-1, 3** **N.Q-1**	**Number & Quantity** **N.RN-1, 3** **1.** Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5.* **3.** Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational. **N.Q-1** **1.** Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.
H.NO.3a comparing the effectiveness of two plausible arguments, distinguishing correct logic or reasoning from that which is flawed, and if there is a flaw in an argument, explaining it **Algebra** **A.REI-1, 2, 5, 11**	**Algebra** **A.REI-1, 2, 5, 11** **1.** Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method. **2.** Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise. **5.** Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions. **11.** Explain why the *x*-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations.

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A.SSE-2,3c **A.CED-3, 4** **A.APR-4, 6**	Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions **A.SSE-2,3c** 2. Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.* 3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. c. Use the properties of exponents to transform expressions for exponential functions. *For example the expression 1.15^t can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.* **A.CED-3, 4** **3.** Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* **4.** Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.* **A.APR-4, 6** **4.** Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples.* **6.** Rewrite simple rational expressions in different forms; write $^{a(x)}/_{b(x)}$ in the form $q(x) + {}^{r(x)}/_{b(x)}$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.
H.NO.3a comparing the effectiveness of two plausible arguments, distinguishing correct logic or reasoning from that which is flawed, and if there is a flaw in an argument, explaining it **Geometry** **G.SRT-4** **G.C-1** **G.CO-9, 10, 11** **G.PE-4**	**Geometry** **G.SRT-4** **4.** Prove theorems about triangles. *Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.* **G.C-1** **1.** Prove that all circles are similar. **G.CO-9, 10, 11** **9.** Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.* **10.** Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.* **11.** Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.* **G.PE-4** **4.** Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, v3) lies on the circle centered at the origin and containing the point (0, 2).*
H.NO.3a comparing the effectiveness of two plausible arguments, distinguishing correct logic or reasoning from that which is flawed, and if there is a flaw in an argument, explaining it **Statistics & Probability** **S.ID-9** **S.IC-5, 6**	**Statistics & Probability** **S.ID-9** **9.** Distinguish between correlation and causation. **S.IC-5, 6** **5.** Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant. **6.** Evaluate reports based on data.

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High School (Grades 9-12) Learning Targets, Progress Indicators, & Common Core Standards **Measurement (ME):** Measurement attributes, processes, and tools help us quantify, compare, and solve problems involving objects, situations, and events. ***H.ME-1*** *Explore measurable attributes, measurement systems, and processes of measurement of more complex or abstract quantities.*	
Progress Indicators for Grades 9-12	**Grades 9-12 CCSS standards** Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★) (CCSS, p. 57).
H.ME.1a making decisions about units and scales that are appropriate for problem-solving situations within or across mathematics disciplines or real-world contexts **N.Q-1, 2** **G.CO-12** **G.MG-1, 2, 3**	**N.Q-1, 2** **1.** Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. **2.** Define appropriate quantities for the purpose of descriptive modeling. **G.CO-12** **12.** Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.* **G.MG-1, 2, 3** **1.** Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder). **2.** Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot) **3.** Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).
H.ME.1b investigating the results when linear dimensions of objects change by some factor (e.g., area and volume change disproportionately: area in proportion to the square of the factor and volume in proportion to its cube) **A.REI-3** **F.FB-1a**	**A.REI-3** **3.** Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters. **F.FB-1a** **1.** Write a function that describes a relationship between two quantities. a. Determine an explicit expression, a recursive process, or steps for calculation from a context.
H.ME.1c exploring quantifications of real-world applications of abstract units such as sound (decibels, pitch), gigabytes, Richter scale, acceleration, and other less tangible units of measure **N.Q-1, 2**	**N.Q-1, 2** **1.** Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. **2.** Define appropriate quantities for the purpose of descriptive modeling.

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High School (Grades 9-12) Learning Targets, Progress Indicators, & Common Core Standards **Measurement (ME):** Measurement attributes, processes, and tools help us quantify, compare, and solve problems involving objects, situations, and events. ***H.ME-2*** *Apply and analyze techniques at an appropriate level of precision and use formulas to quantify or interpret abstract events, objects, and situations.*	
Progress Indicators for Grades 9-12	**Grades 9-12 CCSS standards** Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★) (CCSS, p. 57).
H.ME.2a analyzing levels of precision, accuracy, and approximate error in measurement situations **N.Q-3**	**N.Q-3** **3.** Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.
H.ME.2b using techniques of measurement, estimating, or calculating to compare or analyze two- and three-dimensional figures and their parts **G.SRT-1, 2** **G.C-5** **G.MG-1, 2, 3**	**G.SRT-1, 2** **1.** Verify experimentally the properties of dilations given by a center and a scale factor: a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged. b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor. **2.** Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides. **G.C-5** **5.** Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector. **G.MG-1, 2, 3** **1.** Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder). **2.** Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot) **3.** Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

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High School (Grades 9-12) Learning Targets, Progress Indicators, & Common Core Standards **Patterns, Relations, and Functions (PRF):** Patterns, relations, and functions are used to represent and analyze change in various contexts, make predictions and generalizations, and provide models and explanations for real-world phenomena. ***H.PRF-1*** *Approximate, calculate, model, and interpret change: Use graphical and numerical data resulting from complex situations; Model complex real-world phenomena to make predictions and provide explanations.*	
Progress Indicators for Grades 9-12	**Grades 9-12 CCSS standards** Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★) (CCSS, p. 57).
H.PRF.1a approximating, calculating, and interpreting rates of change using graphical and numerical data **S.ID-1, 2, 7** **F.LE-1b, 1c, 3**	**S.ID-1, 2, 7** **1**. Represent data with plots on the real number line (dot plots, histograms, and box plots). **2.** Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets. **7.** Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data. **F.LE-1b, 1c, 3** **1.** Distinguish between situations that can be modeled with linear functions and with exponential functions. **b.** Recognize situations in which one quantity changes at a constant rate per unit interval relative to another. **c.** Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another. **3.** Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.
H.PRF.1b exploring how the rate of change of something depends on how much there is of something else (as the rate of change of speed is proportional to the amount of force acting) **A.CED-4** **S.ID-3** **F.LE-1b**	**A.CED-4** **4.** Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.* **S.ID-3** **3.** Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers). **F.LE-1b** **1.** Distinguish between situations that can be modeled with linear functions and with exponential functions. **b.** Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.
H.PRF.1c creating mathematical models, using rules and relationships to describe and predict objects and events in the real world **A.CED-2** **S.IC-2** **F.LE-1** **F.TF-5**	**A.CED-2** **2.** Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. **S.IC-2** **2.** Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?* **F.LE-1** **1.** Distinguish between situations that can be modeled with linear functions and with exponential functions. **a.** Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals. **b.** Recognize situations in which one quantity changes at a constant rate per unit interval relative to another. **c.** Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another. **F.TF-5** ***5.*** Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★

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H.PRF.1d recognizing that there limitations in mathematics models **A.CED-3** **A.REI-2** **S.IC-2**	**A.CED-3** **3.** Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* **A.REI-2** **2.** Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise. **S.IC-2** 2. Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.

High School (Grades 9-12) Learning Targets, Progress Indicators, & Common Core Standards

Patterns, Relations, and Functions (PRF): Patterns, relations, and functions are used to represent and analyze change in various contexts, make predictions and generalizations, and provide models and explanations for real-world phenomena.

H.PRF-2 *Use trends and analyze a variety of mathematical patterns, relations, and explicit and recursive functions.*

Progress Indicators for Grades 9-12	**Grades 9-12 CCSS standards** Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★) (CCSS, p. 57).
H.PRF.2a interpreting and rewriting a variety of expressions or functions to solve problems **A.SSE-1, 2, 3, 4** **F.BF-1a, 1b, 2**	**A.SSE-1, 2, 3, 4** **1.** Interpret expressions that represent a quantity in terms of its context.★ a. Interpret parts of an expression, such as terms, factors, and coefficients. b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P.* **2.** Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.* **3.** Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★ a. Factor a quadratic expression to reveal the zeros of the function it defines. b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines. c. Use the properties of exponents to transform expressions for exponential functions. *For example the expression 1.15^t can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.* **F.BF-1a, 1b, 2** **1.** Write a function that describes a relationship between two quantities.★ a. Determine an explicit expression, a recursive process, or steps for calculation from a context. b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* ***2.*** Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★
H.PRF.2b creating equations and inequalities (in one or two variables) and use them to solve problems and graph solutions	**A.CED-1, 2** **1.** Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* **2.** Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. **A-REI-3, 4, 6, 7, 10, 12** **3.** Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

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A.CED-1, 2 **A-REI-3, 4, 6, 7, 10, 12** **S.ID-6**	**4.** Solve quadratic equations in one variable. **a.** Use the method of completing the square to transform any quadratic equation in *x* into an equation of the form $(x – p)2 = q$ that has the same solutions. Derive the quadratic formula from this form. **b.** Solve quadratic equations by inspection (e.g., for $x2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers *a* and *b*. **6.** Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables. **7.** Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line* $y = –3x$ and the circle $x2 + y2 = 3$. **10.** Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line). **12.** Graph the solutions to a linear inequality in two variables as a halfplane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes. **S.ID-6** **6.** Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. **a.** Fit a function to the data; use functions fitted to data to solve problems in the context of the data. *Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.* **b.** Informally assess the fit of a function by plotting and analyzing residuals. **c.** Fit a linear function for a scatter plot that suggests a linear association.
H.PRF. 2c using trends that follow a pattern and are described mathematically to make generalizations or predictions **A-REI-11** **F.BF-3, 4a**	**A-REI-11** **11.** Explain why the *x*-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.⋆ **F.BF-3, 4a** **3.** Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of *k* (both positive and negative); find the value of *k* given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.* **4.** Find inverse functions. **a.** Solve an equation of the form f(x) = c for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) =2 x3$ *or* $f(x) = (x+1)/(x–1)$ *for* $x \neq 1$.
H.PRF. 2d analyzing functions (using technology) by investigating significant characteristics (e.g. intercepts, asymptotes) **A-REI-11** **F.IF-7a, b, e** **F.BF-3** **S.ID-8**	**A-REI-11** **11.** Explain why the *x*-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.⋆ **F.IF-7a, b, c, e** **7.** Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ⋆ **a.** Graph linear and quadratic functions and show intercepts, maxima, and minima. **b.** Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. **e.** Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. **F.BF-3** **3.** Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of *k* (both positive and negative); find the value of *k* given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.* **S.ID-8** **8.** Compute (using technology) and interpret the correlation coefficient of a linear fit.
H.PRF. 2e comparing	**F.IF-9**

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the properties of classes of functions **F.IF-9**	**9.** Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum*.

High School (Grades 9-12) Learning Targets, Progress Indicators, & Common Core Standards **Geometry (GM):** Visualizations, spatial reasoning, and properties of two- and three-dimensional figures can be used to analyze, represent, and model geometric concepts and relationships. ***H.GM-1*** *Explain solutions using geometric attributes and relationships in diverse contexts: Extend understanding of congruence and similarity working with complex figures and situations; Solve problems involving quadrilaterals and triangles; Perform geometric constructions and use informal proofs to describe relationships and transformations.*	
Progress Indicators for Grades 9-12	**Grades 9-12 CCSS standards** Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★) (CCSS, p. 57).
H.GM.1a applying the Pythagorean Theorem **G.SRT-8**	**G.SRT-8** **8.** Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.
H.GM.1b using congruence and similarity relationships to solve problems, including triangle congruence relationships **G.CO-7, 8** **G.SRT-2, 3, 5**	**G.CO-7, 8** **7.** Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent. **8.** Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions. **G.SRT-2, 3, 5** **2.** Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides. **3.** Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar. **5.** Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.
H.GM.1c applying understanding of rotations, reflections, and translations to construct figures (e.g., using coordinates, models, drawings, transparencies, dynamic geometry software) **G.CO-3, 5**	**G.CO-3, 5** **3.** Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself. **5.** Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.
H.GM.1d applying scale factors in solving multiple similarity problems, including transformations in the coordinate plane and similarity relationships with right triangles **G.CO-2** **G.SRT-1b, 2**	**G.CO-2** **2.** Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch). **G.SRT-1b** **1.** Verify experimentally the properties of dilations given by a center and a scale factor: **b.** The dilation of a line segment is longer or shorter in the ratio given by the scale factor. **G.SRT-2** **2.** Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all

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	corresponding pairs of sides.
H.GM.1e making various geometric constructions, including use of dynamic geometry software, and creating informal proofs of relationships (lines and angles, circles, polygons) **G.CO-12, 13**	**G.CO-12, 13** **12.** Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.* **13.** Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.
H.GM.1f solving problems (including proofs) using the relationships among special quadrilaterals (parallelogram, rectangle, rhombus, square, trapezoid, and kite) and describing the characteristics of parallelograms using side, angle, and diagonal properties and relationships **G.CO-9, 10, 11**	**G.CO-9, 10, 11** **9.** Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.* **10.** Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.* **11.** Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.*

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High School (Grades 9-12) Learning Targets, Progress Indicators, & Common Core Standards **Data Analysis, Probability, and Statistics (DPS):** Questions are posed and investigated by collecting data or retrieving existing data, and representing, analyzing, and interpreting data. Investigations, inferences, and predictions are used to make critical and informed decisions. ***H.DPS-1*** *Design and conduct statistical studies: Use appropriate statistical measures for analysis; Develop the concepts of statistical inference and statistical significance, especially in relation to probability principles and sampling distributions.*	
Progress Indicators for Grades 9-12	**Grades 9-12 CCSS standards** Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★) (CCSS, p. 57).
H.DPS.1a designing and conducting different kinds of studies using categorical and numerical data, explain results, and use data to estimate a population mean or proportion: • observational studies (e.g., traffic patterns at an intersection near the school); • sample surveys (a survey of student nutritional habits); • simple comparative experiments (e.g., comparisons of water and fertilizer treatments in a plant growth experiment) **S.IC-3, 4, 5**	**S.IC-3, 4, 5** **3.** Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each. **4.** Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling. **5.** Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.
H.DPS.1b representing data with plots on the real number line (dot plots, histograms, box plots) **S.ID-1**	**S.ID-1** **1.** Represent data with plots on the real number line (dot plots, histograms, and box plots).
H.DPS.1c analyzing and summarizing the data resulting from studies using statistical measures appropriate to shape of the data (median, mean) and spread (interquartile range, standard deviation), and using data to support inferences (population parameters, sample size) or explain possible outliers **S.ID-2, 3, 4, 5** **S.IC-1**	**S.ID-2, 3, 4, 5** **2.** Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets. **3.** Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers). **4.** Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve. **5.** Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data. **S.IC-1** **1.** Understand statistics as a process for making inferences about population parameters based on a random sample from that population.
H.DPS.1d representing and interpreting data (graphs, scatter plots) to explain how variables are related, or to fit a function to the data **S.ID-6**	**S.ID-6** **6.** Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models. b. Informally assess the fit of a function by plotting and analyzing residuals.

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	c. Fit a linear function for a scatter plot that suggests a linear association.

High School (Grades 9-12) Learning Targets, Progress Indicators, & Common Core Standards **Data Analysis, Probability, and Statistics (DPS):** Questions are posed and investigated by collecting data or retrieving existing data, and representing, analyzing, and interpreting data. Investigations, inferences, and predictions are used to make critical and informed decisions. ***H.DPS*-2** *Use the rules of probability to interpret data, develop explanations, and address real-world problems.*	
Progress Indicators for Grades 9-12	**Grades 9-12 CCSS standards** Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★) (CCSS, p. 57).
H.DSP.2a explaining the outcomes of probabilities in words and recognizing equivalent representations of probability, such as one out of ten, 10%, 1/10, 0.10	*No specific Common Core Standards have been linked to this Progress Indicator at this grade level; however, instruction should include these skills/concepts as part of the "hypothesized" learning continuum.*
H.DSP.2b exploring (framing effects) the degree to which we rate something as "good" or "bad"/ "desirable or "undesirable" when numerical information is presented positively (75% lean) or negatively (25% fat)	*No specific Common Core Standards have been linked to this Progress Indicator at this grade level; however, instruction should include these skills/concepts as part of the "hypothesized" learning continuum.*
H.DSP.2c designing and conducting multi-stage (compound) probability experiments (independent events) and comparing the results with theoretical probabilities **S.CP-2**	**S.CP-2** **2.** Understand that two events *A* and *B* are independent if the probability of *A* and *B* occurring together is the product of their probabilities, and use this characterization to determine if they are independent.
H.DSP.2d constructing and interpreting two-way frequency tables when two categories are associated with each object being classified **S.CP-4**	**S.CP-4** **4.** Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.*
H.DSP.2e researching and finding real-world examples and explaining the concept of conditional probability (e.g., compare the chances of having lung cancer if you are a smoker with the chances of being a smoker if you have lung cancer) **S.CP-5**	**S.CP-5** **5.** Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.*

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